Chapter 14

Fuzzy Sets, Fuzzy Systems and Applications

Neural Networks: A Classroom Approach
Satish Kumar
Department of Physics & Computer Science
Dayalbagh Educational Institute (Deemed University)

Important Quotes

- "When using a mathematical model, careful attention must be given to the uncertainties in the model."
  - R. P. Feynman, *On the reliability of the Challenger Space Shuttle*

Important Quotes

- "Everything is vague to a degree you do not realize until you have tried to make it precise."
  - Bertrand Russell, *The Philosophy of Logical Atomism*

Important Quotes

- "Logicians have too much neglected the study of vagueness, not suspecting the important part of it plays in mathematical thought."
  - Charles S. Pierce, *Collected Works*

Important Quotes

- "Fuzzy set theory is wrong, wrong and pernicious…Fuzzy logic is the cocaine of science."
  - William Kahan, *University of California, Berkeley*

The world is gray but science is black and white...

- Logical truth (absolute): $1+1 = 2$

Exists only in math worlds

- Factual truth (vague): "Grass is green"

Ubiquitous in the real world

The Fuzzy Principle

- The FUZZY PRINCIPLE states that

  Everything is a matter of degree

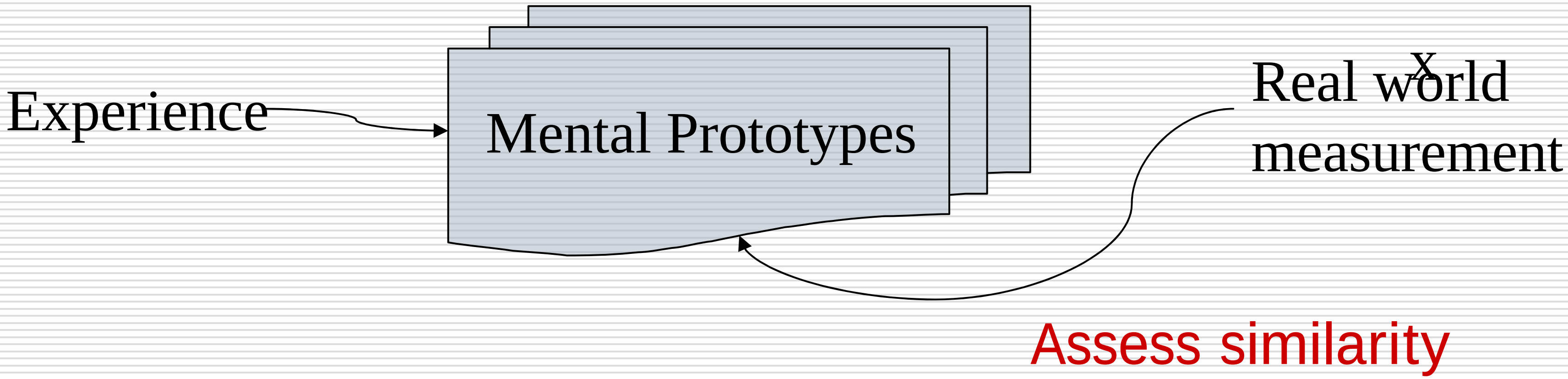
- Fuzziness stems from lexical imprecision - **an elasticity in the meaning of real world concepts**

Humans reason with *fuzzy* concepts: Example (Bezdek, 1993)

- Advice to driving student:
  - Begin braking 74 feet from the crosswalk
  - Apply the brakes *pretty soon*
- Children quickly learn to interpret:
  - You must be in bed *around* 9 pm
- We assimilate imprecise information and vague rules and reason effortlessly using a ***fuzzy logic***

What are fuzzy sets ?

- Is a 40 year old MIDDLE-AGED ?
- Is a 50 year old MIDDLE-AGED ?

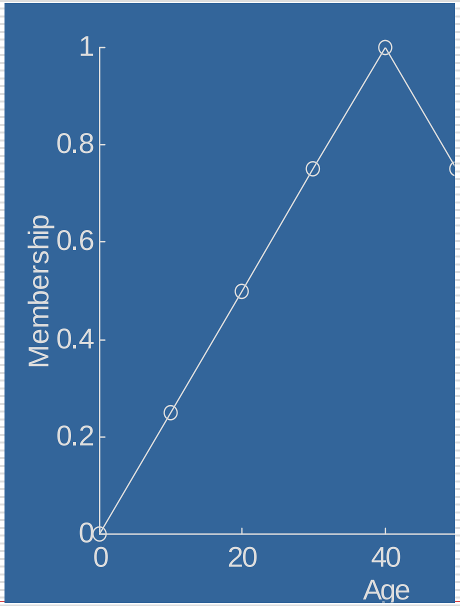


- What we try to do is to assess the *compatibility* or *similarity* of x with our *mental prototype*

What are fuzzy sets ? (contd.)

- Alternatively: Given that you are MIDDLE-AGED what is the *possibility* that you are aged x years?

Age	Possibility/Belief/Compatibility
0	0.0
10	0.25
20	0.5
30	0.75
40	1.0
50	0.75
60	0.5
70	0.25
80	0



What are fuzzy sets ? (contd.)

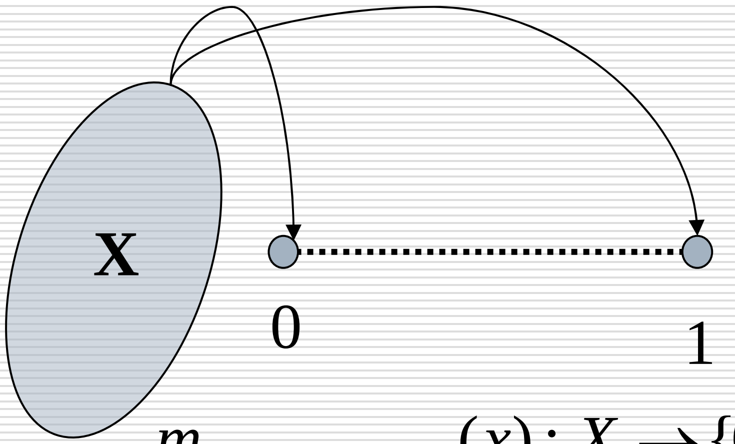
Mathematically we have a *membership function* :

$$\mu_{MIDDLE-AGED}(x) : X \rightarrow [0,1]$$

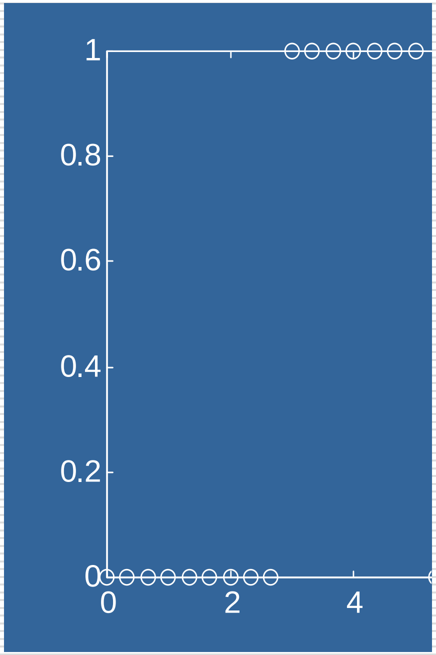
where X is the universe of discourse (UOD).

Classical sets and Fuzzy sets

- Example: Numbers, z, $3 \le z \le 5$

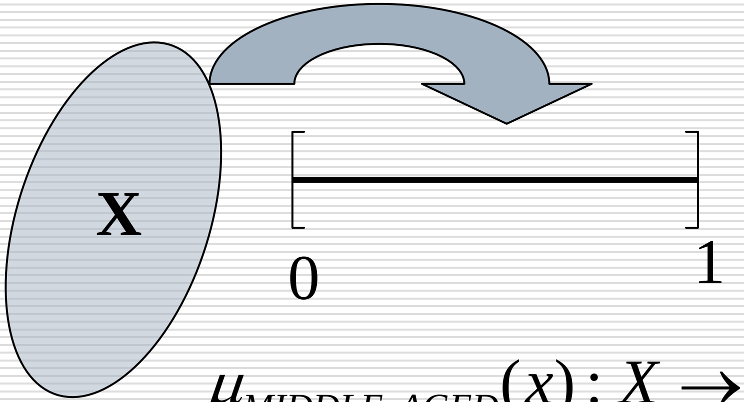


$$m_{MIDDLE-AGED}(x) : X \to \{0,1\}$$

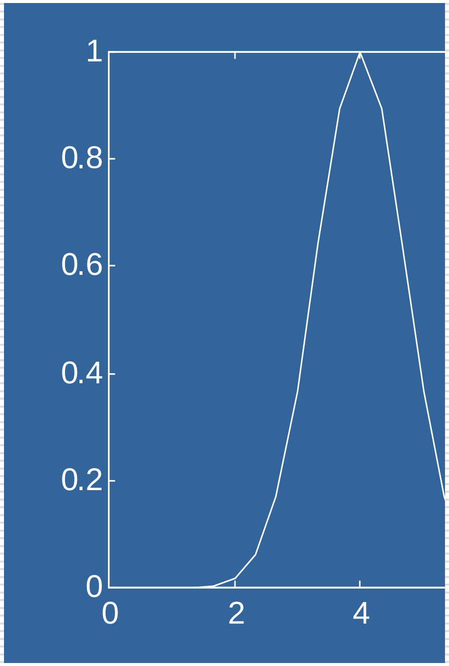


Classical sets and Fuzzy sets

- Example: Numbers *close to 4*



$$\mu_{MIDDLE-AGED}(x) : X \rightarrow [0,1]$$



Important Points to Note…

- Fuzzy sets admit a continuum of memberships
- Classical sets admit only two-state membership
- Crisp sets are unique
- The Gaussian fuzzy set was chosen from an infinite number of possible membership functions.
- To quote Jim Bezdek:
  - **"Uniqueness is sacrificed (and mathematicians howl), but flexibility is increased (and engineers smile)."**

What does the fuzzy power set F(2X) look like ?

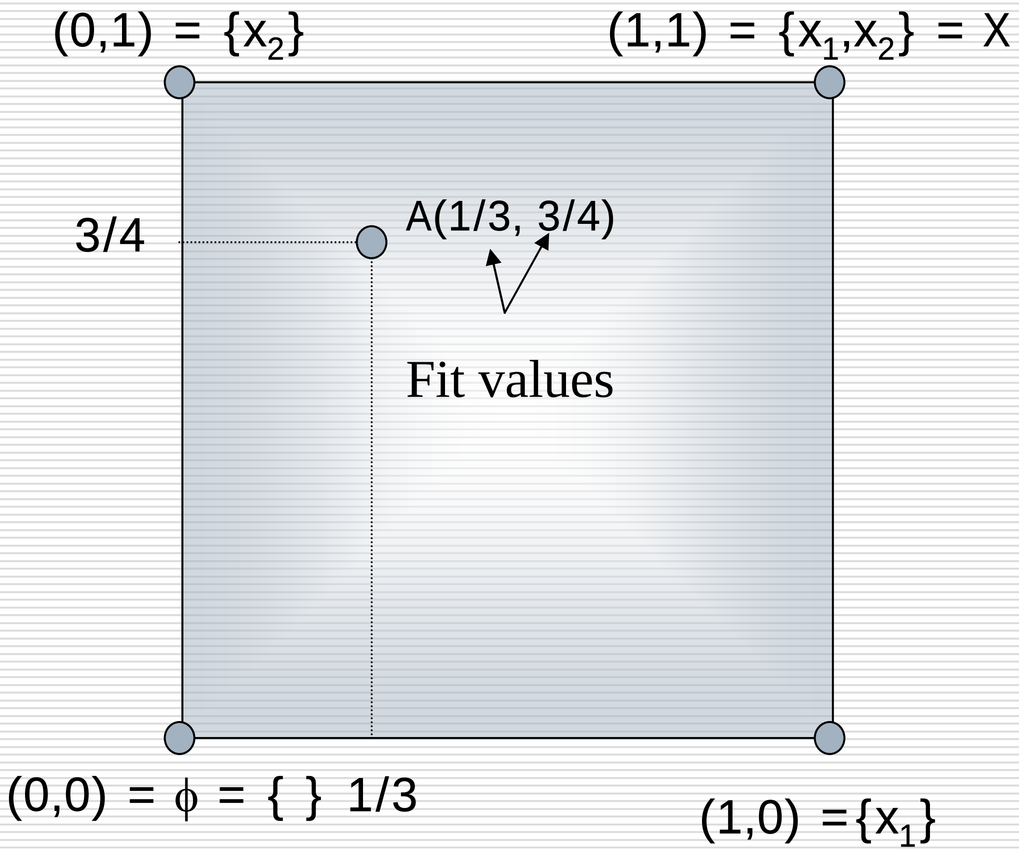
- Example: $X = \{x_1, x_2\}$

$$2^X = \{\phi, \{x_1\}, \{x_2\}, X\}$$

$$(0,0) \quad (1,0) \quad (0,1) \quad (1,1)$$

where coordinates are memberships.

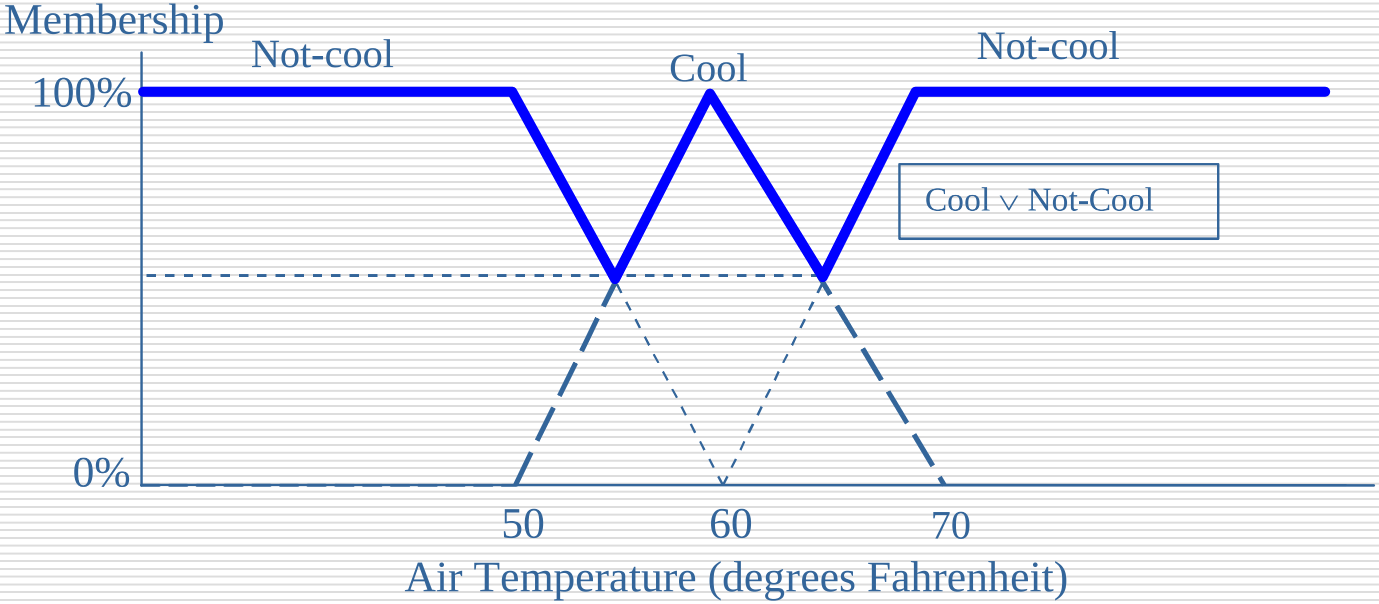
What does the fuzzy power set F(2X) look like ?



- $B^n=\{0,1\}^n = 2X$
- $I^n=[0,1]^n = F(2X)$
- Vertices of I^n are *non-fuzzy* sets
- Fuzzy sets fill in B^n to produce I^n

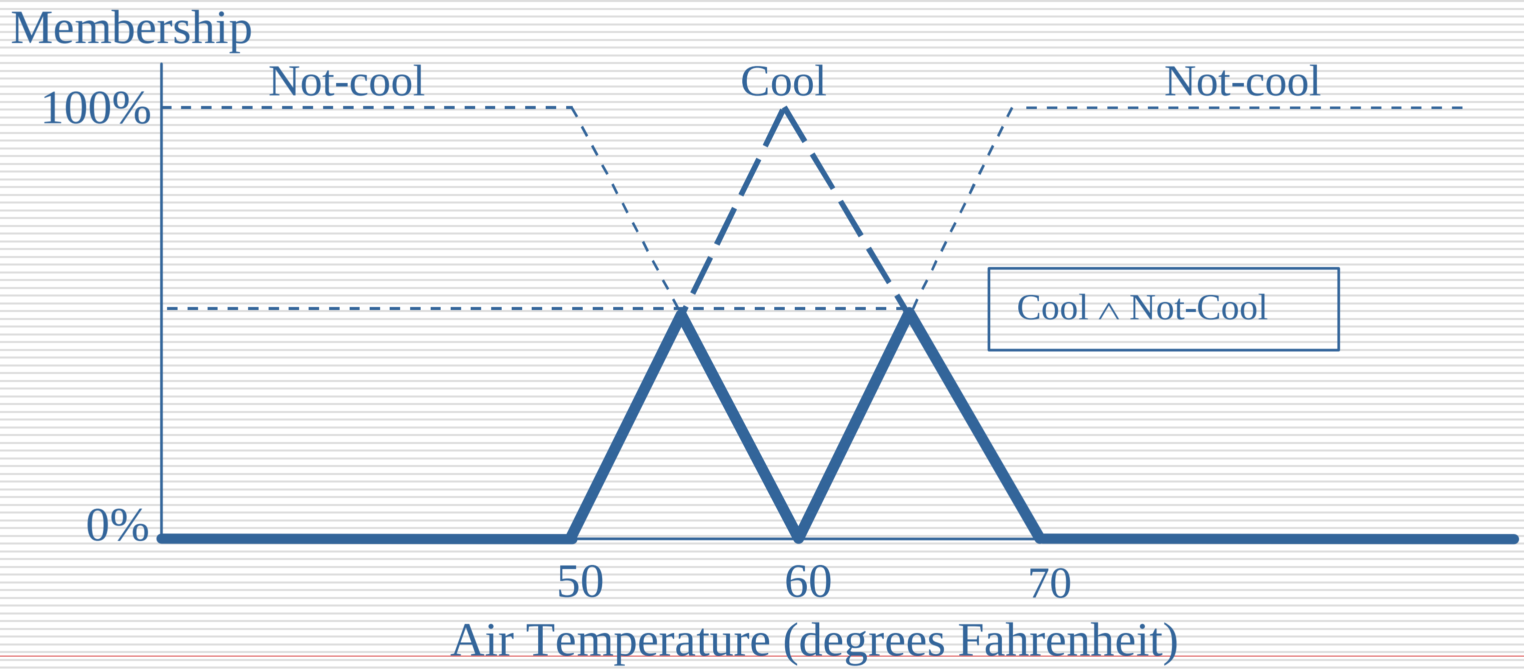
Fuzzy Set Union

Define: $\mu_{A \cup B} = \max(\mu_A, \mu_B)$



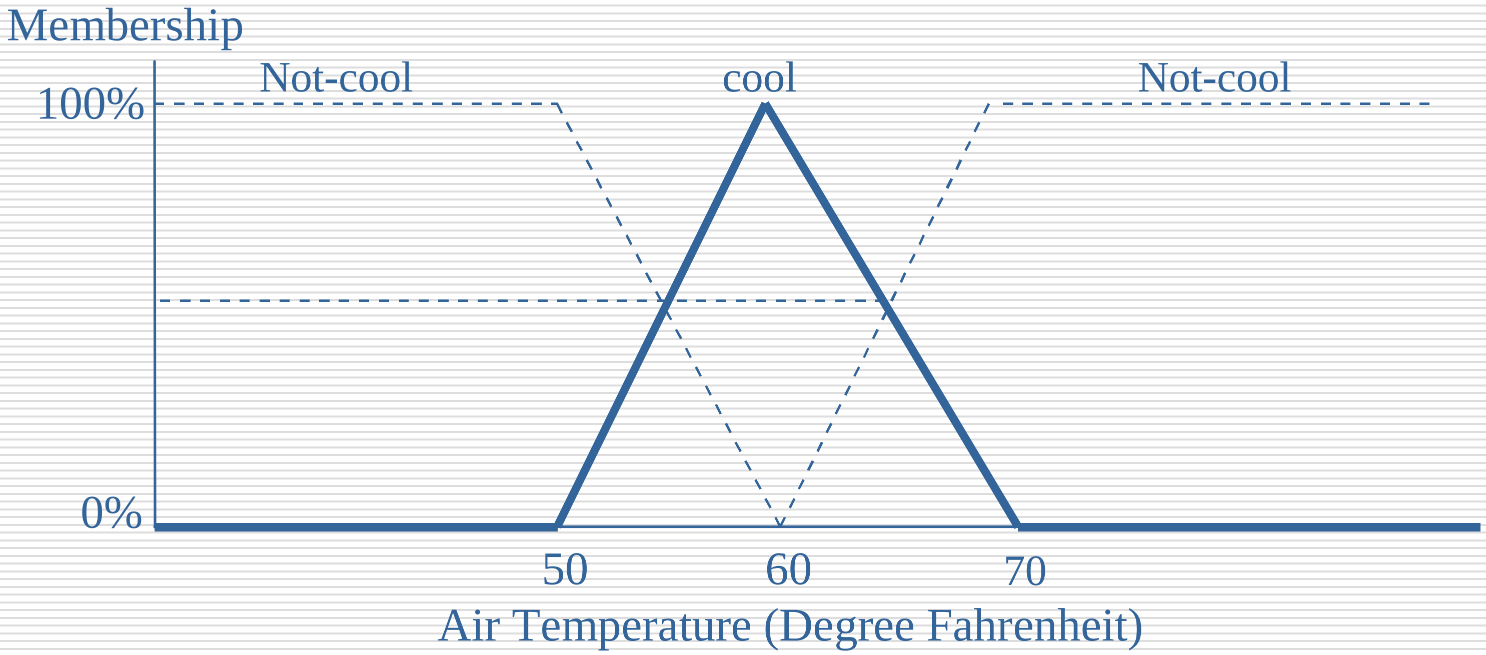
Fuzzy Set Intersection

Define: $\mu_{A \cap B} = \min(\mu_A, \mu_B)$

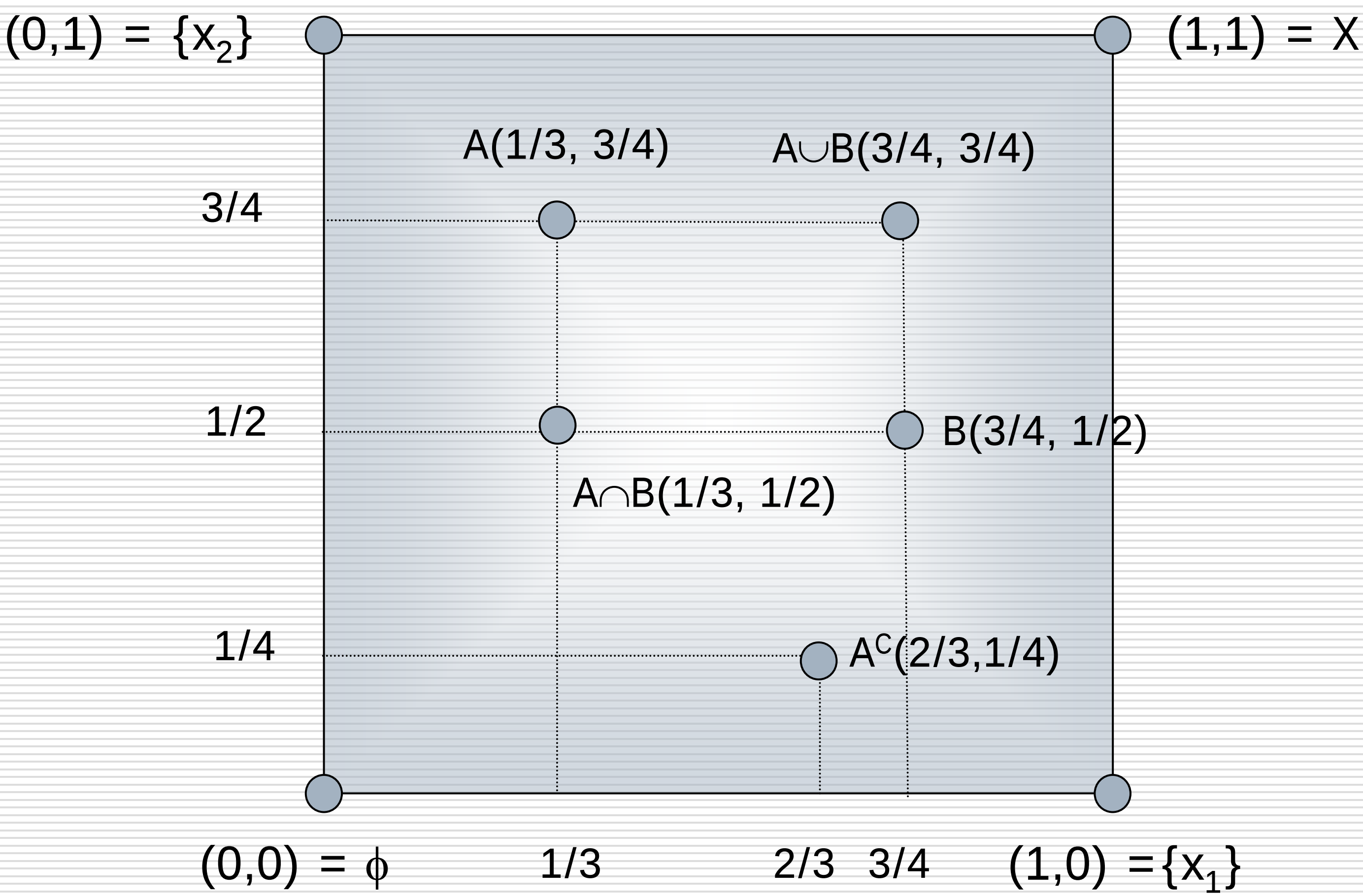


Fuzzy Set Complement

Define: $\mu_{A'} = 1 - \mu_A$



The Geometry of Operations on Fuzzy Sets



Fuzzy Set Operations: Examples

Fuzzy Union

$$A = (1.0 \;\; 0.8 \;\; 0.4 \;\; 0.5)$$
$$B = (0.9 \;\; 0.4 \;\; 0.0 \;\; 0.7)$$
$$A \cup B = (1.0 \;\; 0.8 \;\; 0.4 \;\; 0.7)$$

Fuzzy Intersection

$$A = (1.0 \;\; 0.8 \;\; 0.4 \;\; 0.5)$$
$$B = (0.9 \;\; 0.4 \;\; 0.0 \;\; 0.7)$$
$$A \cap B = (0.9 \;\; 0.4 \;\; 0.0 \;\; 0.5)$$

Fuzzy Complement

$$A = (1.0 \;\; 0.8 \;\; 0.4 \;\; 0.5) \quad B = (0.9 \;\; 0.4 \;\; 0.0 \;\; 0.7)$$
$$A^C = (0.0 \;\; 0.2 \;\; 0.6 \;\; 0.5) \quad B^C = (0.1 \;\; 0.6 \;\; 1.0 \;\; 0.3)$$

Fuzzy Operations hold for Classical Sets too!

$A = (1\ \ 0\ \ 1\ \ 1\ \ 0) = \{x_1, x_3, x_4\}$

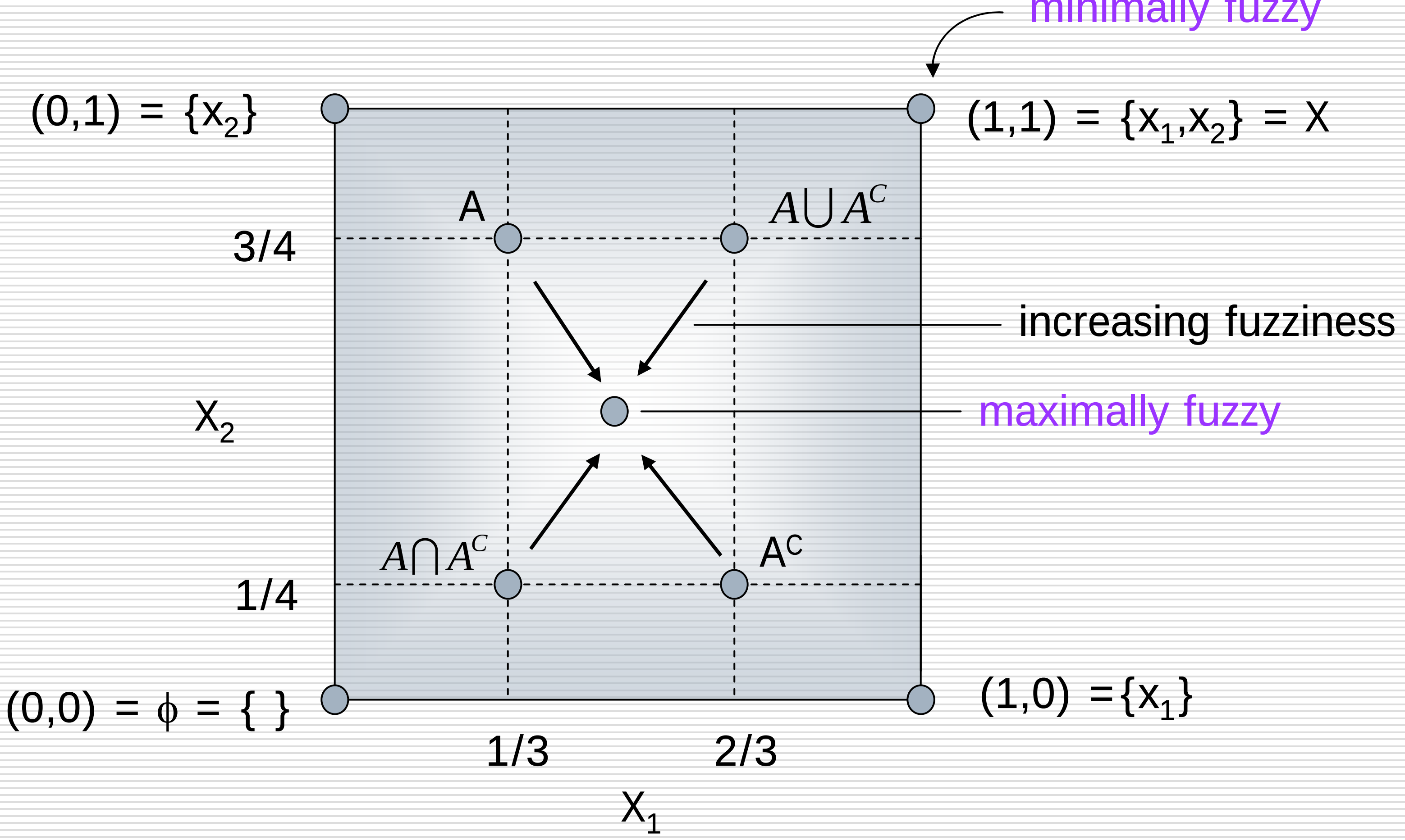
$B = (1\ \ 1\ \ 1\ \ 0\ \ 0) = \{x_1, x_2, x_3\}$

$A \cap B = (1\ 0\ 1\ 0\ 0) = \{x_1, x_3\}$

$A \cup B = (1\ 1\ 1\ 1\ 0) = \{x_1, x_2, x_3, x_4\}$

$A^C = (0\ 1\ 0\ 0\ 1) = \{x_2, x_5\}$

Completing The Fuzzy Square



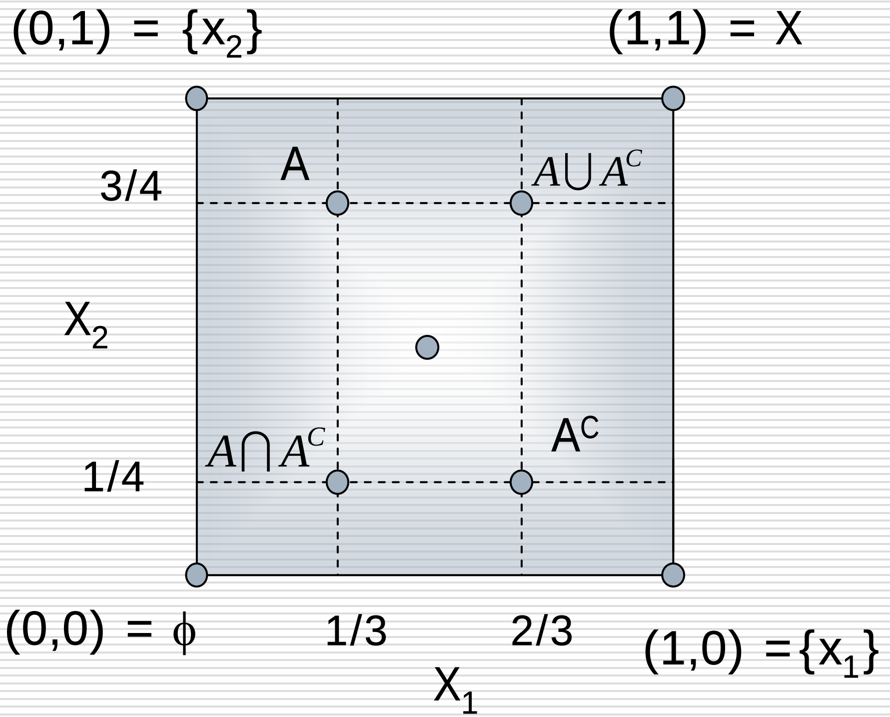
Completing The Fuzzy Square (contd.)

Proposition:

A is properly fuzzy
iff $A \cap A^C \neq \phi$
iff $A \cup A^C \neq X$

At the midpoint:

$$A = A^C = A \cap A^C = A \cup A^C$$



Fuzzy Cube Midpoint Solves the Classical Logical Paradox

- The Cretan said: "All Cretan's are liars"
- California bumper sticker: "Don't trust me"
- The barber shaves those who don't shave themselves.
  - **Who shaves the barber?**

Completing The Fuzzy Square (contd.)

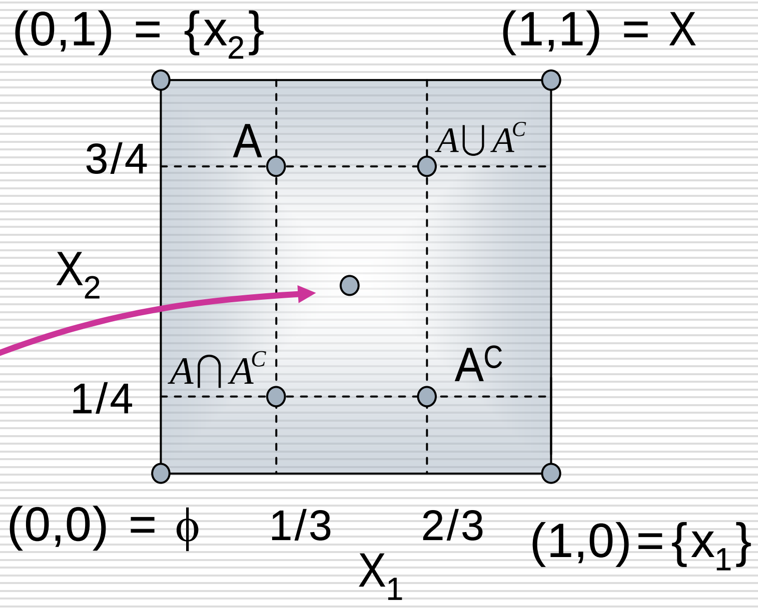
Paradox At The Midpoint

S $\Rightarrow$ not (S) and not (S) $\Rightarrow$ S

or t(S) = not (t(S))

= 1 – t(S)

$\Rightarrow$ t(S) = 0.5



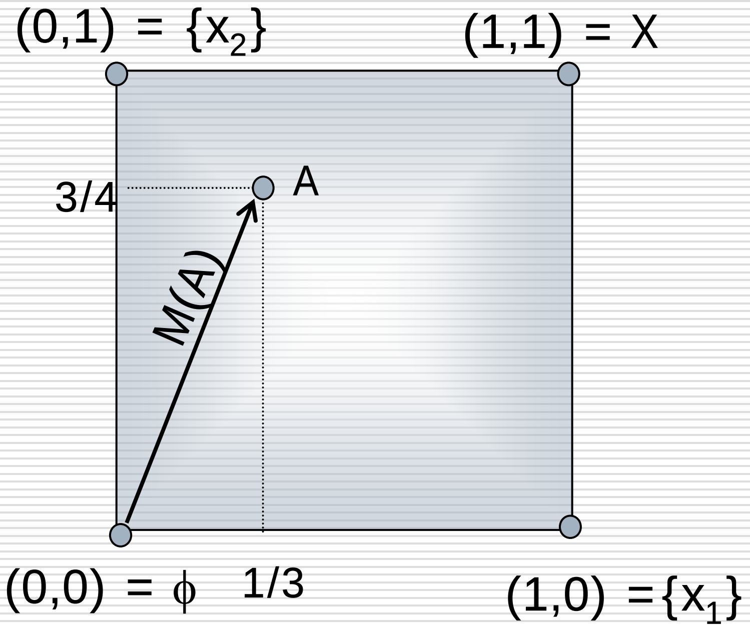
$$A = A^C = A \cap A^C = A \cup A^C$$

How big is a fuzzy set?

The sigma count of a fuzzy set A, M(A), is the sum of its fit values:

$$M(A) = \sum_{i=1}^{n} \mu_A(x_i)$$

and is an l^1 distance.



$$A = \left(\frac{1}{3}, \frac{3}{4}\right) \rightarrow \quad M(A) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}.$$

How big is a fuzzy set ?

The sigma count of a fuzzy set A, M(A), is the sum of its fit values and is an l^1 distance, where the l^p distance between fuzzy sets A and B in I^n is :

$$l^p(A,B) = \sqrt[p]{\sum_{i=1}^{n} \left|\mu_A(x_i) - \mu_B(x_i)\right|^p} \quad 1 \le p \le \infty$$

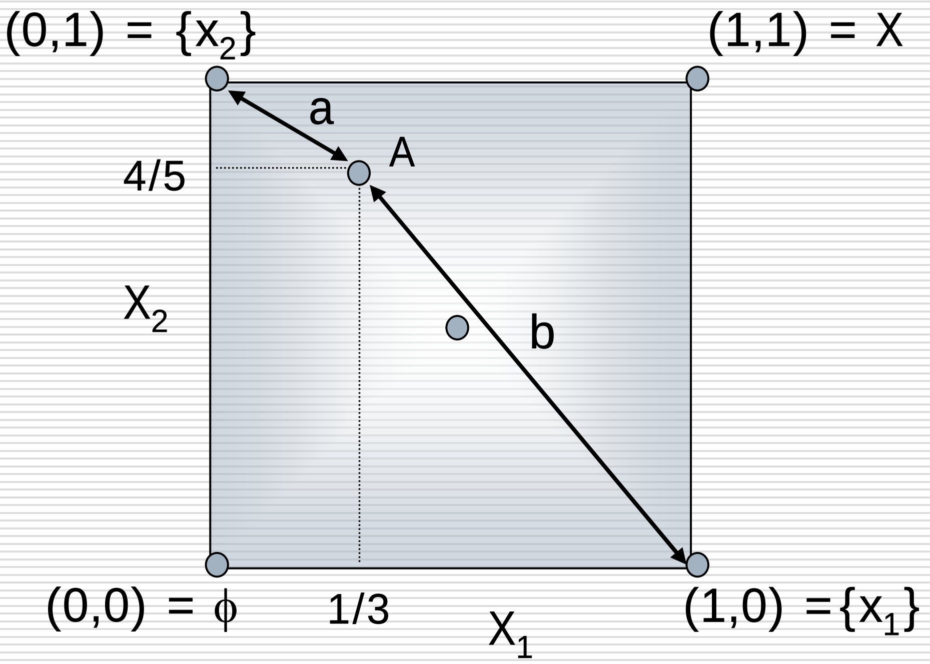
$$\begin{aligned} M(A) &= \sum_{i=1}^{n} \mu_A(x_i) \\ &= \sum_{i=1}^{n} \left|\mu_A(x_i) - 0\right| \\ &= \sum_{i=1}^{n} \left|\mu_A(x_i) - \mu_\phi(x_i)\right| \\ &= l^1(A,\phi) \end{aligned}$$

Sigma counts become classical cardinalities at the corners of I^n

How *fuzzy* is a fuzzy set ?

We measure the entropy of a fuzzy set:

$$E(A) = \frac{a}{b} = \frac{l^1(A, A_{near})}{l^1(A, A_{far})}$$



$$l^1(A, A_{near}) = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

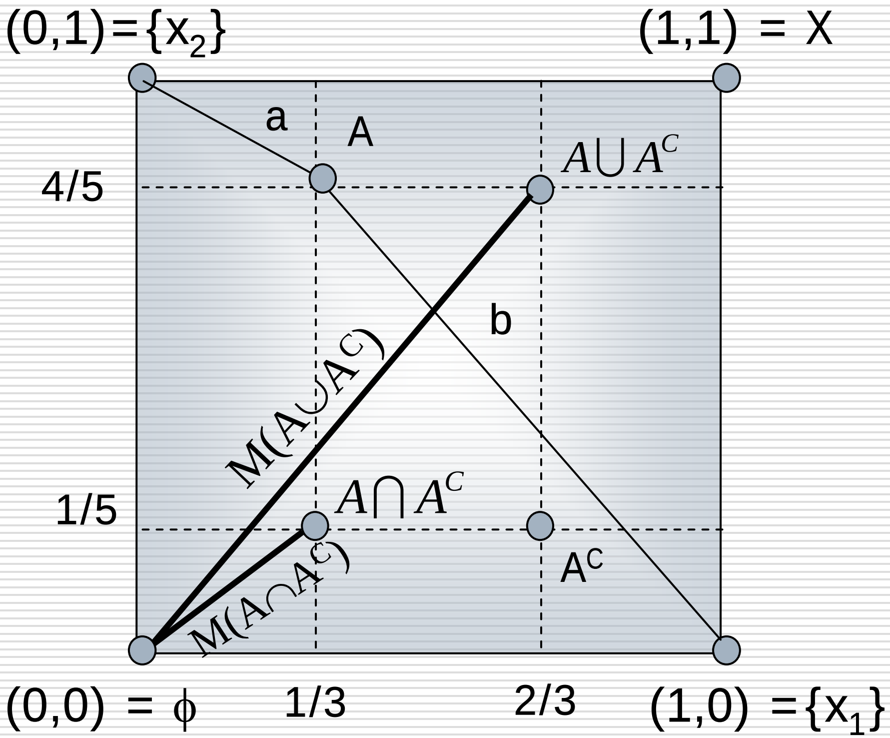
$$l^1(A, A_{far}) = \frac{2}{3} + \frac{4}{5} = \frac{22}{15}$$

$$E(A) = \frac{8/15}{22/15} = \frac{8}{22}$$

The Fuzzy Entropy Theorem

Fuzzy Entropy Theorem:

$$E(A) = \frac{a}{b} = \frac{M(A \cap A^C)}{M(A \cup A^C)}$$



At any of the corners E(A) = 0

At the midpoint E(A) = 1

Subsets

- Classically

$$A \subset B \ \text{ iff } \ A \in 2^B$$

- Representing sets as bivalent functions or "indicator" functions

$$m_A(x) \to \{0,1\}$$

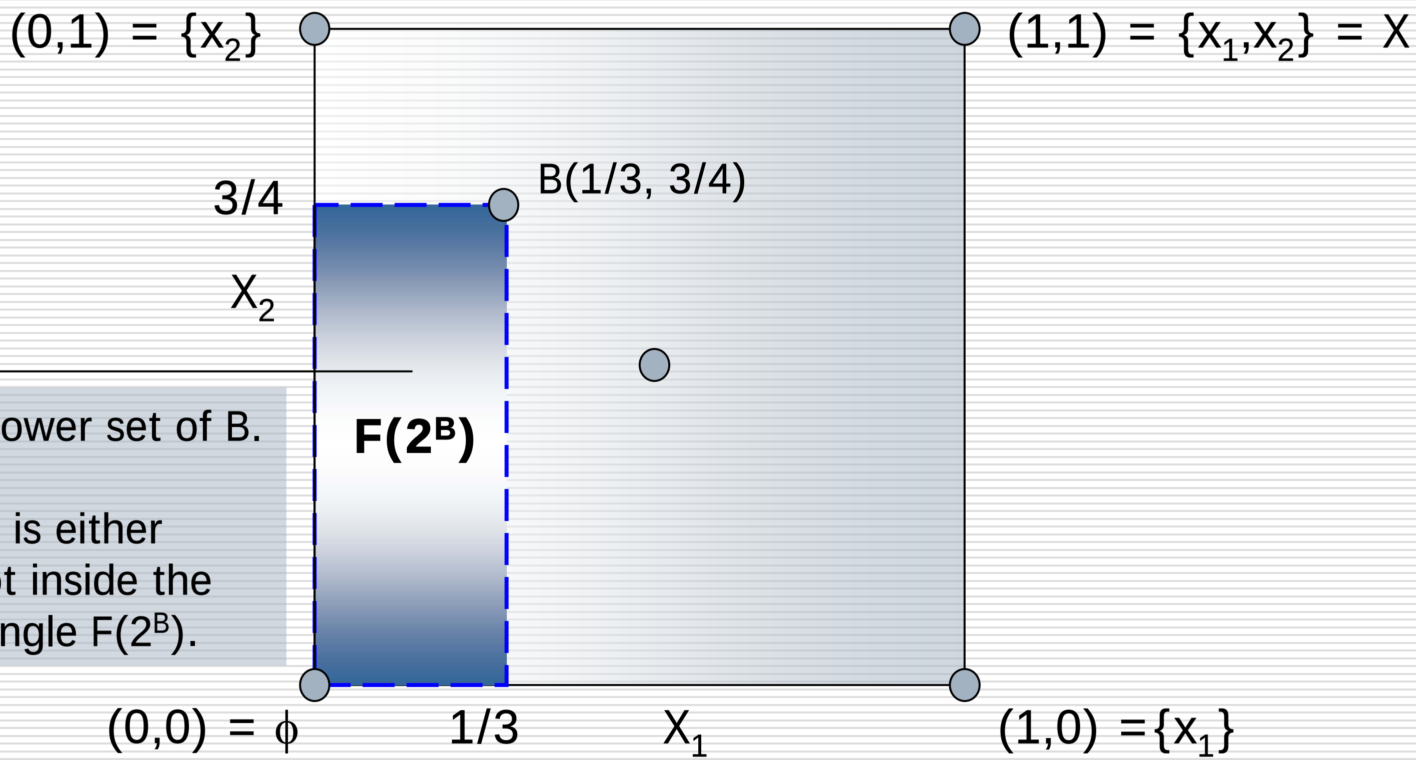
- A $\subset$ B iff there is no element x that belongs to A but not to B:

$$m_A(x) = 1 \text{ and } m_B(x) = 0$$

Fuzzy Subsets - Dominated Membership

A⊂ B if and only if $\mu_A(x) \le \mu_B(x) \quad \forall x \in X$

This is non-fuzzy



The fuzzy power set of B.

Any point A is either inside or not inside the hyper-rectangle $F(2^B)$.

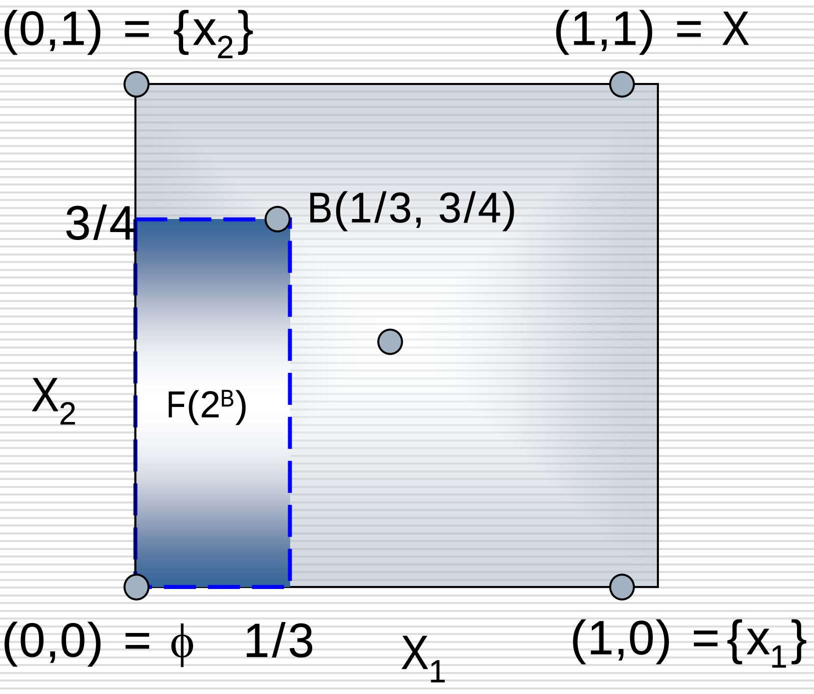
Subsethood

Different points outside the hyperrectangle $F(2^B)$ resemble subsets of B to different degrees.

$$S(A, B) = \text{degree}(A \subset B)$$

$$= \mu_{F(2^B)}(A)$$

$S(\bullet,\bullet) \in [0,1]$.



If X has a thousand elements and only one element violates the dominated membership function, A is overwhelmingly a subset of B.

Subsethood: Algebraic Derivation

Fit violation strategy: Count fit violations in magnitude and frequency

SUPERSETHOOD = 1 – S(A, B)

Count violations by adding them:

$$\sum_{x \in X} \max(0, \mu_A(x) - \mu_B(x))$$

Subsethood: Algebraic Derivation (contd.)

$$\text{Supersethood} = \frac{\sum_{x \in X} \max\left(0, \mu_A(x) - \mu_B(x)\right)}{M(A)}$$

$$S(A,B) = 1 - \frac{\sum_{x \in X} \max(0, \mu_A(x) - \mu_B(x))}{M(A)} \qquad 0 \le S(A,B) \le 1$$

S(A, B) = 1 iff $\mu_A(x_i) \le \mu_B(x_i) \quad \forall\, x_i \in X$

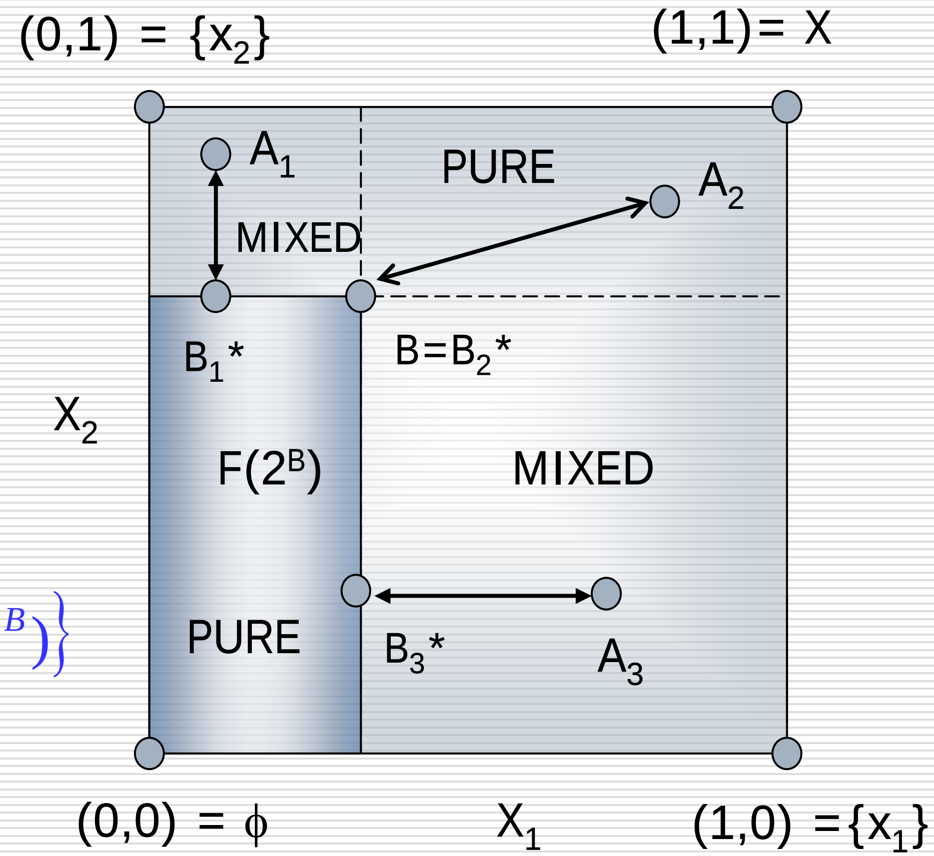
S(A, B) = 0 iff $B = \phi$

The empty set has no subsets

Subsethood: Geometric Derivation

- How close is A to $F(2^B)$? The fuzzy power set extends to slice I^n into 2n hyperrectangles

$$d(A, F(2^B)) = \inf\{d(A, B') \mid B' \in F(2^B)\}$$
$$= d(A, B^*)$$



Subsethood:Geometric derivation

A natural interpretation defines supersethood as:

$$d(A, F(2^B)) = d(A, B^*)$$

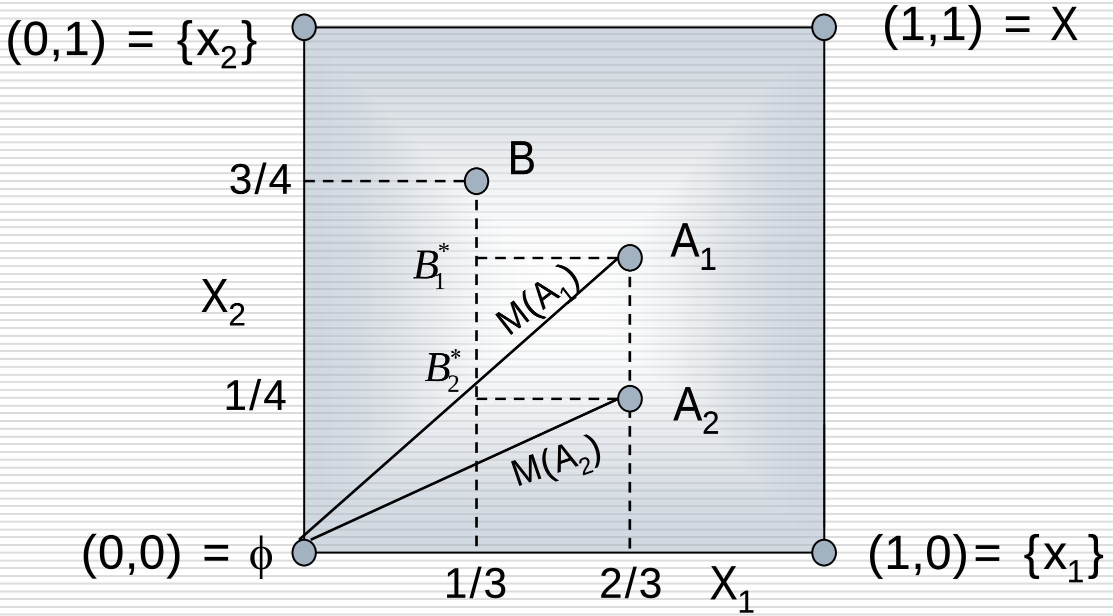
Normalization by M(A) yields:

$$S(A,B) = 1 - \frac{d(A,B^*)}{M(A)} = 1 - \frac{\sum_{i=1}^{n} \max(0, \mu_A(x_i) - \mu_B(x_i))}{M(A)}$$

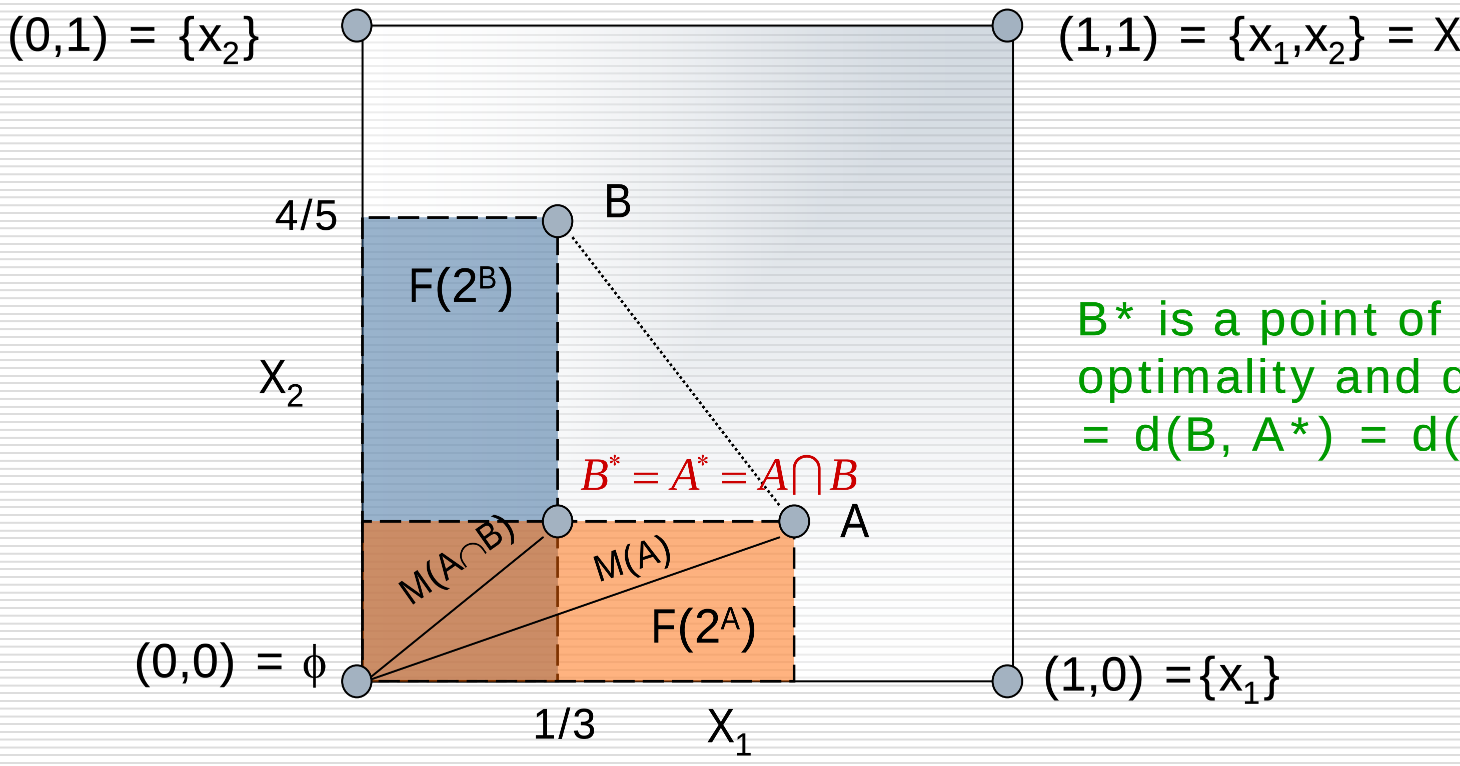
Subsethood: Geometric derivation

$A_2 = (0.66, 0.25)$
$B \;= (0.33, 0.75)$
$B_2^* = (0.33, 0.25)$

$\mu_{A2} - \mu_B = (0.33, -0.5)$
$\max(0, \mu_{A2} - \mu_B) = (0.33, 0)$
$d(A_2, B^*) = 0.33 + 0 = 0.33$



Subsethood Theorem



B* is a point of double optimality and d(B, F(2A)) = d(B, A*) = d(B, B*).

Since d(A, B*) = M(A) - M(A∩B) : $S(A,B) = 1 - \frac{d(A,B^*)}{M(A)} = \frac{M(A \bigcap B)}{M(A)}$

Entropy-Subsethood Theorem

Theorem: $E(A) = S(A \cup A^C, A \cap A^C)$

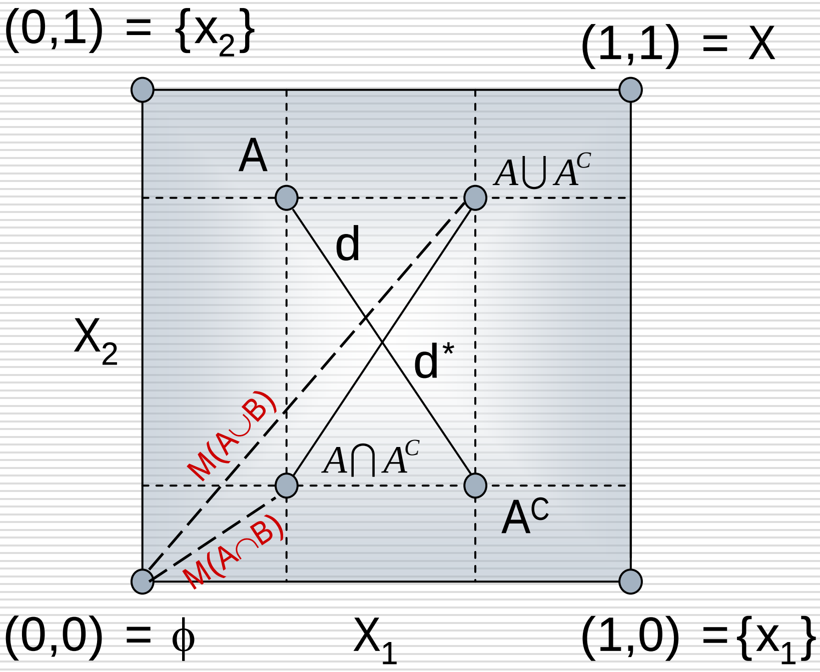
Proof:

$$S(A,B) = \frac{M(A \cap B)}{M(A)}$$

$$S(A \cup A^C, A \cap A^C)$$

$$= \frac{M((A \cup A^C) \cap (A \cap A^C))}{M(A \cup A^C)}$$

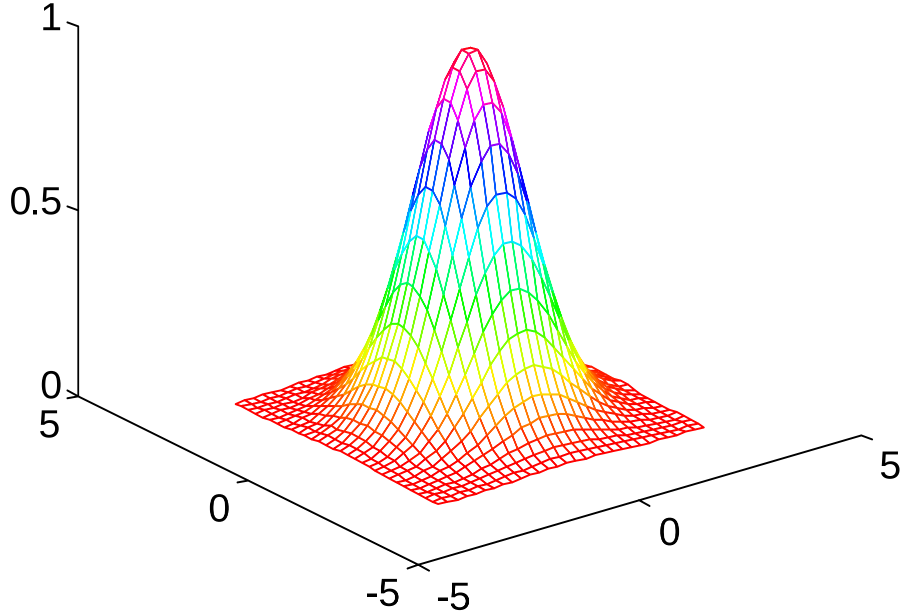
$$= \frac{M(A \cap A^C)}{M(A \cup A^C)} = E(A)$$



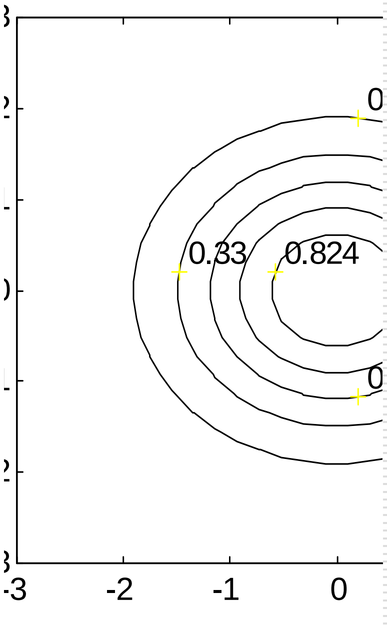
A Simple Approximation Example

$$z(x,y) = e^{-0.5(x^2+y^2)}$$

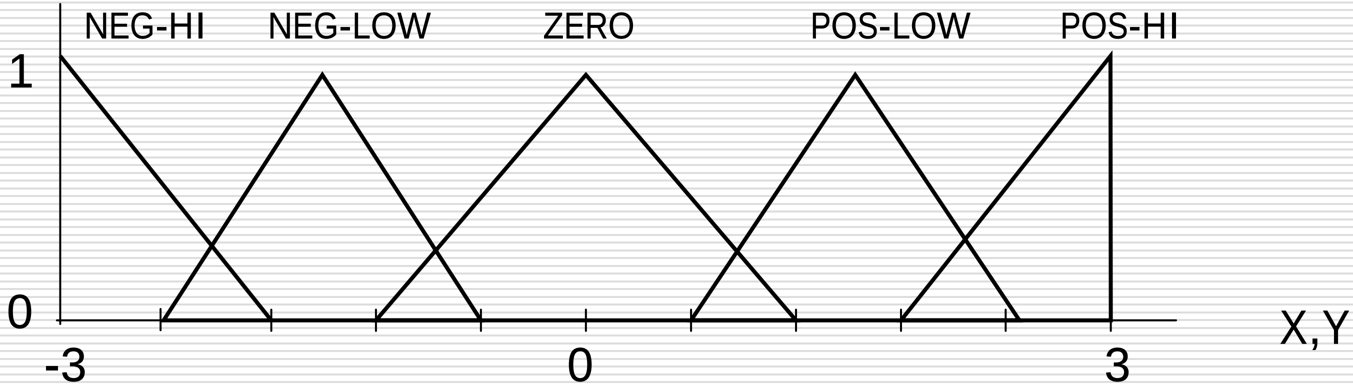
3-d mesh plot

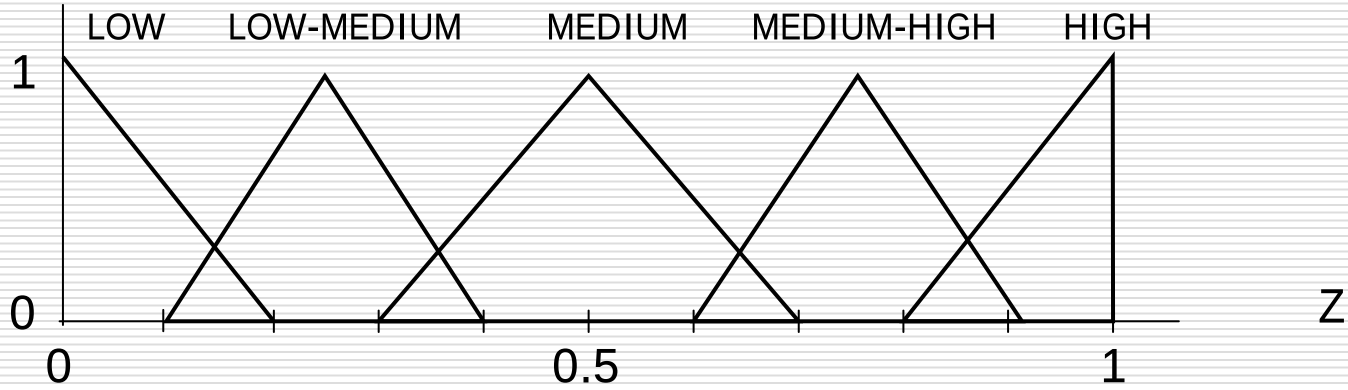


Contour plot

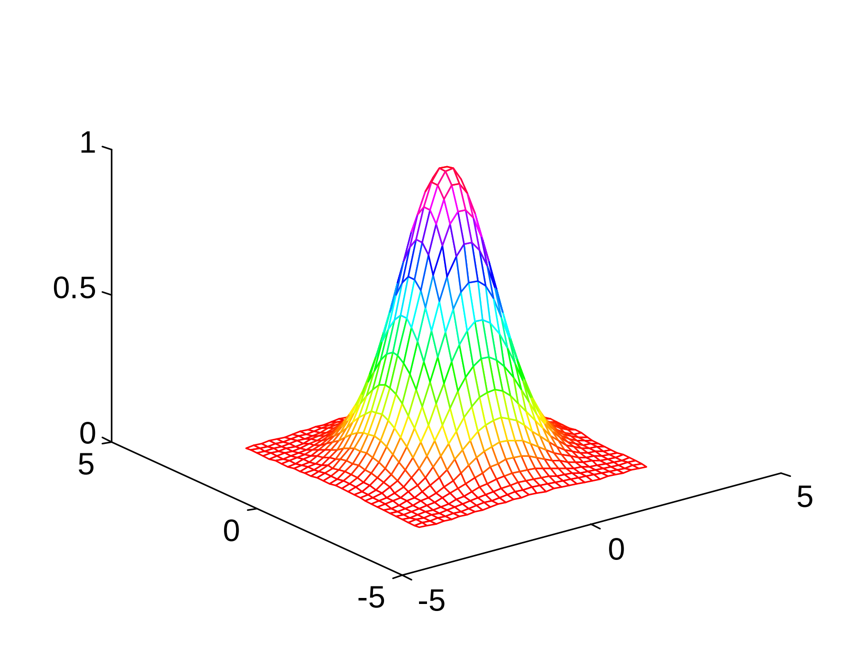


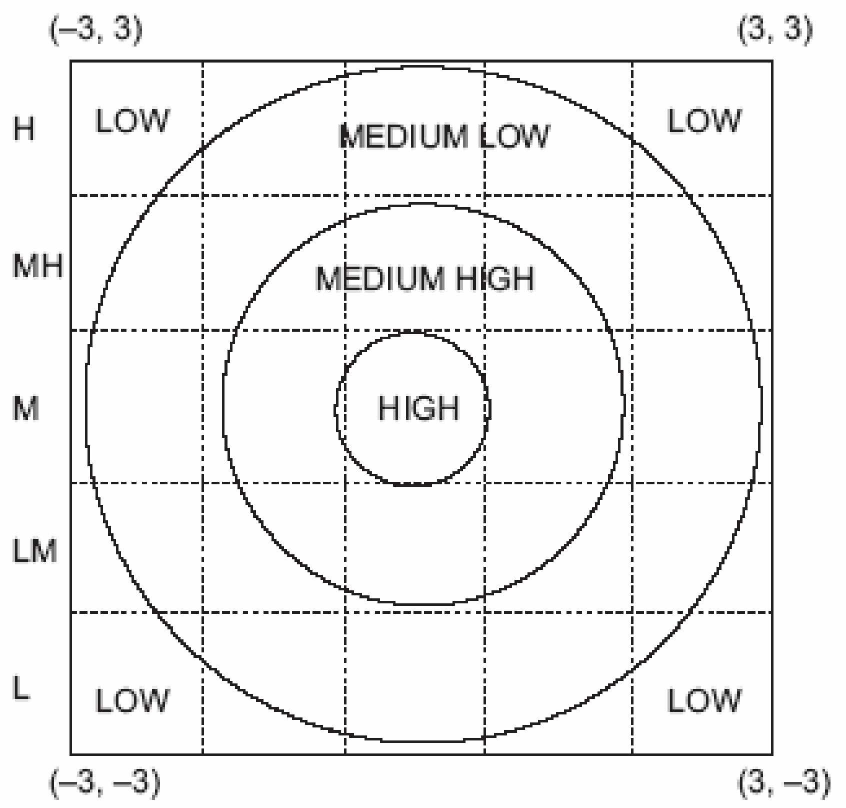
Defining fuzzy sets on the UODs





Layout Rules on the Contour Plot

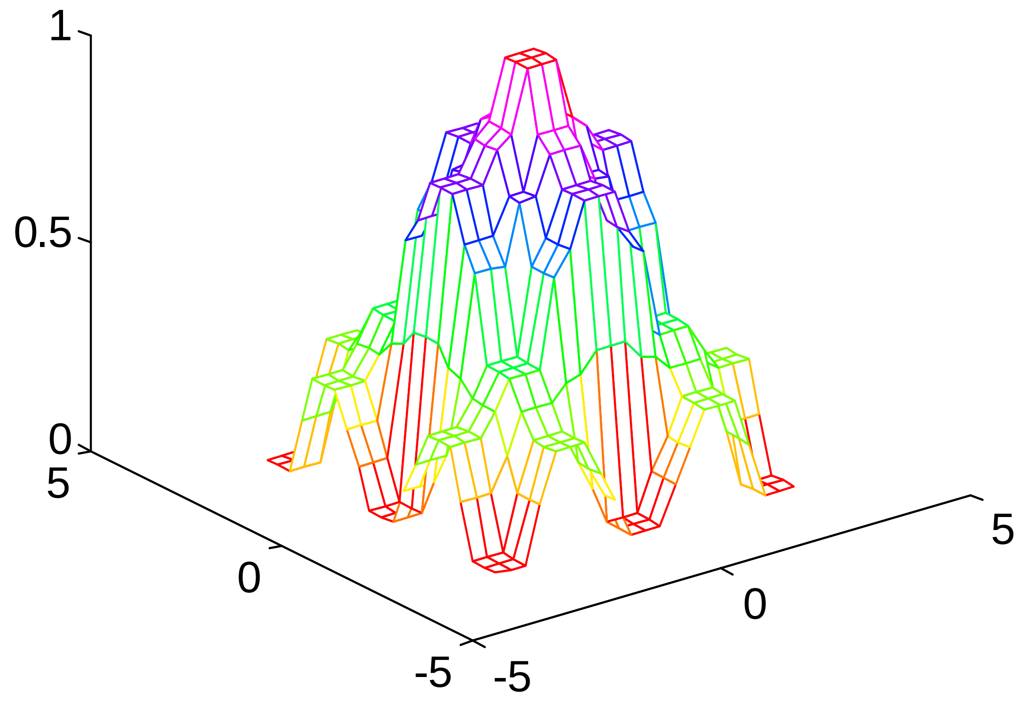




Developing the First Cut Rule Base

POS-HI	LOW	LOW	MED-LOW	LOW	LOW
POS-LOW	LOW	MED	MED-HI	MED	LOW
ZERO	MED-LOW	MED-HI	HIGH	MED-HI	MED-LOW
NEG-LOW	LOW	MED	MED-HI	MED	LOW
NEG-HI	LOW	LOW	MED-LOW	LOW	LOW
X↑ Y→	NEG-HI	NEG-LOW	ZERO	POS-LOW	POS-HI

25 Rule Base Simulation



49 Rule Base Simulation